Proximity Drawability: A Survey *
(extended abstract)

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Abstract. Increasing attention has been given recently to drawings of graphs in which edges connect vertices based on some notion of *proximity*. Among such drawings are Gabriel, relative neighborhood, Delaunay, sphere of influence, and minimum spanning drawings. This paper attempts to survey the work that has been done to date on proximity drawings, along with some of the problems which remain open in this area.

1 Proximity Drawings

In 1969, Gabriel and Sokal [15] presented a method for associating a graph to a set of geographic data points P by connecting points $x, y \in P$ with an edge if and only if the closed disk having the segment $\overline{xy}$ as diameter contained no other point of P. This graph, now called the *Gabriel graph* of P, is just one example of what have come to be called *proximity graphs*. Loosely speaking, a proximity graph is a graph constructed from a set P of points in some metric space by connecting pairs of points which are deemed to be "sufficiently" close together. A set P can give rise to a variety of different proximity graphs depending upon the definition of closeness used. Early work in this area was concerned for the most part with the problems of determining notions of proximity which might best capture the "internal structure" of a set of points and, having done so, of efficiently computing the proximity graph of a given set of points. For a survey of such results, see Jaromczyk and Toussaint [21].

More recently, increasing attention has been given to the *proximity drawing problem*: given a graph G and a definition of proximity, determine whether a set P of points exists such that the proximity graph of P is the given graph, and if so, compute such a set. Clearly the set P, if it exists, gives rise to a straight-line

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drawing of G, called a *proximity drawing* of G, where each vertex of G is mapped to a distinct point of P and each edge to a straight-line segment between pairs of points of P. Proximity drawings have several interesting features. They are usually unaffected by changes in scale, since the measures of proximity used are based on relative distances between points. Also, adjacent vertices are drawn (relatively) more closely together than non-adjacent vertices, and vertices not incident to a particular edge are not drawn too close to the edge. Furthermore, neighbors of a given vertex tend to cluster together. This paper aims to give a survey of the central problems and results in this area. Although many of the ideas described here can be developed in the more general setting of a metric space, we assume that the drawings are to be made in Euclidean d-space; the vast majority of work in this area has concentrated on drawings in the plane.

Many notions of proximity have been proposed and investigated over the past several years. Instead of organizing our presentation historically, however, we will organize it hierarchically around some common features of most proximity drawings, in particular the unifying concept of what we call a *(k, n)-proximity drawing.* See Figure 1 for a (non-proximity) drawing of the proximity drawing hierarchy. With the two notable exceptions of minimum spanning drawings and sphere of influence drawings, which will be discussed later, the proximity drawings which have been studied to date are all examples of (k, n)-proximity drawings and are based on the notion of "proximity regions" or "regions of influence" (even sphere of influence graphs are based on proximity regions; the only difference, as we will soon see, is the way in which the regions are used to define the drawing).

Let R be a function which associates to every set S of $k \geq 2$ points in Euclidean d-space E^d a subset $R(S)$ of E^d; $R(S)$ is called the *proximity region* or *region of influence* of S. Now consider a straight-line drawing Γ of G in which the vertices are drawn at a set of locations P. (Throughout the paper, unless stated otherwise, P will denote the set of vertices of a straight-line drawing of some graph G.) We call Γ a *(k, n)-proximity drawing* of G if Γ is the drawing resulting from the following procedure: For every set $S \subset P$ of k vertices, edges are drawn between all pairs of vertices in S if the proximity region $R(S)$ contains at most n vertices from $P - S$. While the proximity region can be any subset of the space in question, usually the regions chosen are homeomorphic to an open or closed ball of dimension equal to that of the space. Such drawings are referred to as *open* or *closed* proximity drawings, respectively. Some examples of (k, n)-proximity drawings follow.

The *Gabriel region* [15] of two vertices x and y is defined to be the closed sphere (in d dimensions) having the segment $\overline{xy}$ as diameter. A *Gabriel drawing* of G is a straight-line drawing of G having the property that two vertices x and y of the drawing form an edge if and only if the Gabriel region of x and y does not contain any other vertex. Gabriel drawings are an example of a closed *$(2, 0)$-proximity drawing.* Figure 2(a) shows a Gabriel drawing of a planar triangulated graph. The dotted circles in the Figure represent the proximity regions. A *Delaunay drawing* [7] is an example of a closed $(3, 0)$-proximity drawing: here

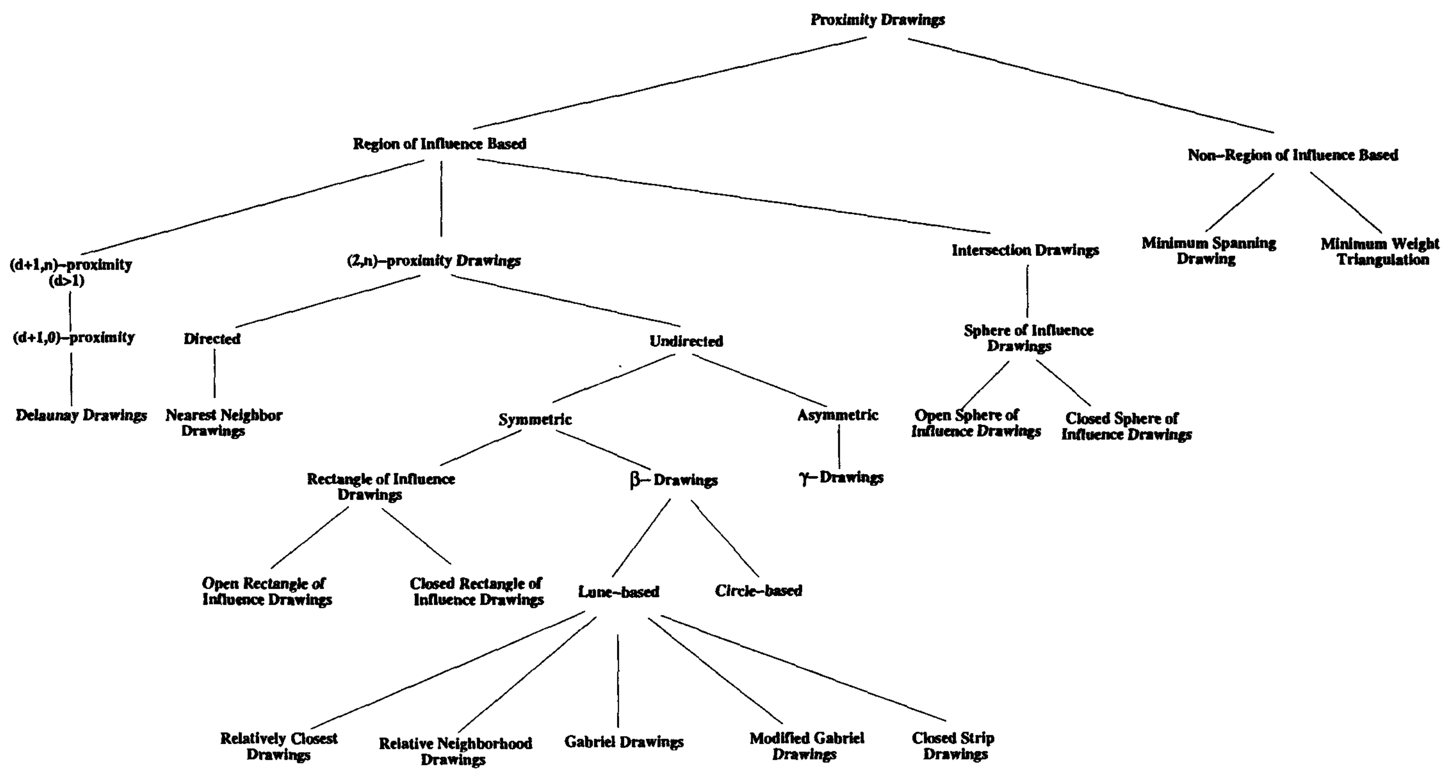

Fig. 1. A taxonomy of proximity drawings. Here d refers to the dimension of the space.

triplets of points in P are connected if the closed disk they determine contains no other points of P. Clearly, Delaunay drawings makes sense only for planar triangulated graphs. A Delaunay drawing of K_4 is shownin Figure 2 (b). Observe that K_4 does not have a Gabriel drawing.

A *relative neighborhood drawing* of a graph G is an open $(2,0)$-proximity drawing in which the proximity region, called the *relative neighborhood region* [33], of two points x and y is the intersection of the open disks of radius $d(x,y)$ centered at x and y. Thus, in a proximity drawing of G, x and y are adjacent if there is no vertex whose distance to both x and y is less than the distance between x and y. A relative neighborhood drawing of a tree consisting of a vertex of degree five adjacent to five leaves is depicted in Figure 2 (c). Note that the same tree does not admit a Gabriel drawing. A Gabriel drawing of a tree is given in Figure 2 (d).

The regions which give rise to Gabriel, Delaunay and relative neighborhood drawings can be modified by changing the requirement that they be empty to the requirement that they contain at most n other vertices. The resulting drawings are referred to as n-Gabriel, n-Delaunay and n-relative neighborhood drawings [31, 32, 1].

In our definition of (k,n)-proximity drawings, we have required that the sets S to which we associate proximity regions contain at least two points, since otherwise no edges can be formed. There is, however, a way in which proximity regions associated with single points can be used to create proximity drawings: pairs of points can be connected by an edge if the regions corresponding to the points intersect. We call such a drawing an *intersection drawing.* An example of such a proximity drawing would be a *sphere of influence drawing* of a graph. To produce this type of drawing, each point $x \in P$ has as its proximity region, the *sphere of influence* [34], the disk centered at x of radius $r_x = \min\{d(x,y) : y \in P - \{x\}\}$. One can consider either open or closed sphere of influence drawings.

While to date, work in proximity drawings has dealt exclusively with problems of drawing undirected graphs, we note that the $(2,n)$-proximity drawing paradigm can also be used to produce drawings of directed graphs by associating with each *ordered* pair of points (x,y) a proximity region $R_{x,y}$. By allowing the region $R_{x,y}$ to be different from the region $R_{y,x}$, it is possible to produce drawings where the edge (x,y) is in the drawing, but not the edge (y,x). An example of this is the *directed nearest neighbor drawing* [28], where each vertex $x \in P$ is connected to all vertices (or sometimes just one) of minimum distance from x. Although the nearest neighbor drawing is usually considered to be an undirected graph, the definition is inherently that of a directed graph. The proximity region $R_{x,y}$ in this case is the open disk of radius $d(x,y)$ centered at x.

Returning now to proximity drawings of undirected graphs, and in particular to $(2,n)$-proximity drawings, we introduce families of open and closed proximity drawings which include several of the drawings mentioned so far. The Gabriel, modified Gabriel, relative neighborhood and relatively closest drawings studied in [26], [25], [3], [5], [35] are all examples of members of a family of drawings called *β-drawings.* In 1985, Kirkpatrick and Radke [23, 30] introduced a family

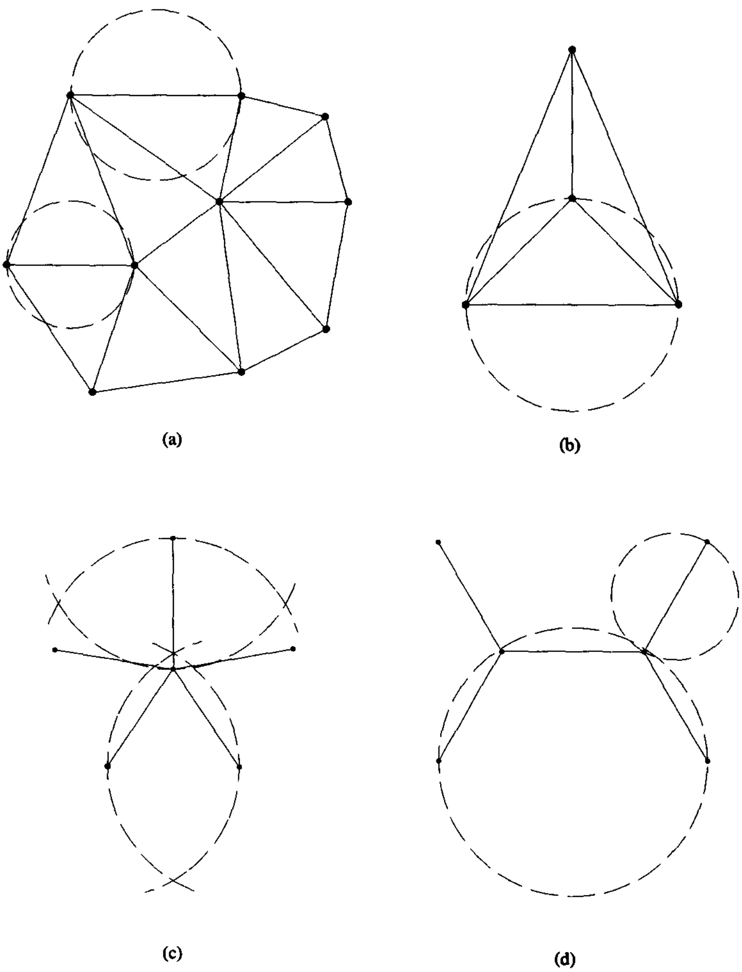

Fig. 2. Four different proximity drawings: (a) A Gabriel drawing of a triangulation, (b) A Delaunay drawing, (c) A relative neighborhood drawing of a tree, (d)A Gabriel drawing of a tree.

of closed $(2,0)$-proximity regions called *β-neighborhoods*, denoted by $R[x,y,\beta]$ and defined as follows (see also Figure 3):

1. For $\beta = 0$, $R[x,y,\beta]$ is the line segment $\overline{xy}$.
2. For $0 < \beta < 1$, $R[x,y,\beta]$ is the intersection of the two closed disks of radius $d(x,y)/(2\beta)$ passing through both x and y.
3. For $1 \leq \beta < \infty$, $R[x,y,\beta]$ is the intersection of the two closed disks of radius $\beta d(x,y)/2$ and centered on the line through x and y.

4. For $\beta = \infty$, $R[x, y, \beta]$ is the closed infinite strip perpendicular to the line segment $\overline{xy}$.

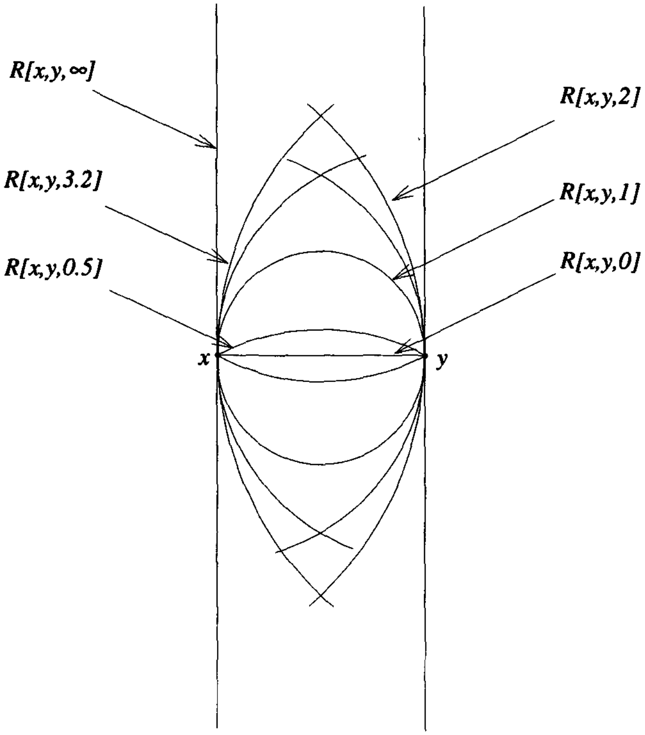

Fig. 3. A set of (2, 0)-proximity regions $R[x, y, \beta]$.

Obviously, one can also define the analogous regions $R(x, y, \beta)$ using open sets instead of closed sets ($R(x, y, 0)$ is defined to be the empty set). The Gabriel, modified Gabriel, relative neighborhood and relatively closest drawings mentioned above are obtained, respectively, from the β-regions $R[x, y, 1]$, $R(x, y, 1)$, $R(x, y, 2)$ and, $R[x, y, 2]$ respectively. The *closed strip drawings* are β-drawings which use the region $R[x, y, \infty]$. Similarly, the *open strip drawings* are β-drawings which use the region $R(x, y, \infty)$. The regions defined above are referred to as *lune-based* β-regions. In the same paper, the authors also describe *circle-based* β-regions: for each $\beta \geq 1$, the region associated with two vertices x and y is the union of the two disks of radius $\beta d(x, y)/2$ passing through both x and y.

From the graph drawing perspective, the central problem is that of determining, for a given graph G, the values of β such that G admits a β-drawing. For example, for $\beta < 2$, only connected graphs admit β-drawings; for $\beta > 1$, only planar graphs do. Since a graph may, for a given value of β, have many—very different—β-drawings, it is also of interest to examine the "aesthetic" properties of such drawings. In [3, 2], several results concerning β-drawings of trees are presented.

More recently, a proximity region called the *rectangle of influence* [19] has been studied from the graph drawing perspective. Here the proximity region associated with two points x and y is the axis-parallel rectangle determined by x and y. As in the case of β-drawings, one can use either open or closed rectangles; as with β-regions, the choice will determine which graphs can be drawn. We note that the drawings determined by a set of points using this region are also of interest in the study of visibility graphs [18, 6, 29].

In defining $(2,0)$-drawings, the proximity region chosen for a pair of vertices x, y is almost always symmetric about the perpendicular bisector of the segment $\overline{xy}$. This guarantees a certain symmetry in the drawings produced. This symmetry, however, is not always desirable. Veltkamp [36] introduced a family of proximity graphs, called *γ-graphs*, in which the proximity region lacks this symmetry. He takes advantage of this absence of symmetry in constructing object boundaries from a set of points.

We close this section by mentioning the one type of proximity drawing which does not seem to arise from any particular proximity region: minimum spanning drawings. A *minimum spanning tree* of a set P of points is a connected, straight-line drawing with vertex set P which minimizes total edge length. A *minimum spanning drawing* of a tree G is a straight-line drawing of G such that, letting P denote the set of locations at which the vertices are drawn, the drawing is a minimum spanning tree of P. A *minimum weight triangulation* of a set P is a triangulation of P having minimum total edge length. A *minimum weight drawing* of a triangulated planar graph G is a straight-line drawing of G such that, letting P denote the set of locations at which the vertices are drawn, the drawing is a minimum spanning triangulation of P. These two types of drawings have strong connections to the proximity drawings that have been discussed here. Some of these connections will be discussed in the next section.

2 Results and Problems

In this section we present a summary of the main results and problems on proximity drawability. Table 2 (see also Fig. 1) lists several types of proximity drawings along with the most common classes of graphs whose drawability properties have been studied. A citation in an entry of the table means that a complete characterization exists describing which graphs in the class admit a proximity drawing of the specified type. A citation followed by the letter P means that only a partial characterization has been given. A dash means that nothing is yet known. An X means that the entry is not meaningful. Finally, a citation followed by the letter M in the column on outerplanar graphs means that the result holds for maximal outerplanar graphs.

		Trees	Outerplanar	Triangulated	Planar	Non-planar
Delaunay		X	[10] M	[11, 12, 9] P	X	X
Minimum Weight		[27, 13]	—	—	X	X
β-drawing	Modified Gabriel	[3]	[25] M	—	[5] P	—
	Gabriel	[3]	[25] M	—	[26] P	[26]
	Relative Neighb.	[3]	[25]	—	[35] P	[33]
	Relatively Closest	[3]	[25]	[5] P	[5] P	[33]
	Open Strip	[2] P	—	[23]	—	[23]
	Closed Strip	[2]	[2]	[2]	[2]	[2]
	General β	[2] P	—	—	—	—
Open Sphere		[20]	—	—	—	—
Closed Sphere		[20]	—	—	—	—
Open Rectangle		[14]	[14]	[14] P	[14] P	—
Closed Rectangle		[14]	[14] M,P	—	[14] P	—

Table 1. Summary of Characterization Results. For explanation, see text.

Observe that the table contains several holes, each one corresponding to a pair family of drawings–family of graphs whose relationships have not been understood yet. In our opinion all of them address meaningful research problems. In particular, triangulated graphs, appear to be one of the most challenging families to investigate.

Before going through the description of the entries of the table, we mention here two other problems that, in our opinion, are worthy of investigation. The first problem concerns the study of (k, n)-proximity drawings for $n > 0$, and is simply to establish combinatorial properties of these drawings. The second problem deals with proximity drawing in three dimensions. The greater availability of workstations and packages for 3D drawing on one side and the increase of the complexity of drawings on the other, strongly suggest that characterizations and algorithms for three-dimensional proximity drawing be investigated. Up to now only very preliminary contributions, have been given in this fascinating area.

2.1 Delaunay Drawings

An exact characterization of those graphs which admit Delaunay drawings has yet to be found; however, several results have been obtained. Delaunay drawability of planar triangulations was studied in [11, 12] where sufficient conditions were given. In particular, the conditions presented in [11] are based on the *toughness* of the graph to be represented. In [10] it was shown that all maximal outerplanar graphs admit Delaunay drawings. In [9] angles of Delaunay drawings were characterized.

2.2 Minimum Spanning Drawings

The problem of deciding whether a tree admits a minimum spanning drawing has been well studied, and is, essentially, solved. Monma and Suri [27] proved that each tree with maximum vertex degree at most 5 can be drawn as a minimum spanning tree of some set of vertices. In the same paper it is shown that no tree having at least one vertex with degree greater than 6 can be drawn as a minimum spanning tree. As for trees having maximum degree equal to six, Eades and Whitesides [13] have recently shown that it is NP-hard to decide whether such trees can be drawn as minimum spanning trees.

Much less is known about the problem of constructing a minimum weight drawing of a planar triangulation. In fact it is still not known whether the problem of constructing a minimum weight triangulation for a given set of points is NP-complete [17]. An interesting connection between minimum weight triangulations and (circle-based) β-drawings was given by Keil [22]. He showed that, given a set P of points, the circle-based $\sqrt{2}$-drawing that has P as the set of vertices is a subgraph of the minimum weight triangulation of P.

2.3 β-Drawings

Kirkpatrick and Radke [23] in 1985 defined the open and closed β-regions ($R(x,y,\beta)$, $R[x,y,\beta]$) discussed in the previous section. They called the graphs that can be drawn with such regions *β-skeletons.* They also listed several applications of such graphs. As mentioned previously, the open and closed β-drawings include a number of well-studied proximity drawings, including Gabriel, Modified Gabriel, relative neighborhood and relatively closest drawings. Matula and Sokal [26], Cimikowski [5], and Urquhart [35] have studied the problems of producing drawings of each of these types for particular classes of graphs including cycles, wheels, trees, and bipartite graphs. Toussaint [33] studied the relationship between the graphs produced by relative neighborhood drawings and other proximity drawings. He showed that the relative neighborhood drawing on a set P of points is a supergraph of every minimum spanning tree of P and a subgraph of the Delaunay triangulation of P. Agarwal and Matoušek [1] showed that the number of edges of an n-vertex graph that has a relative neighborhood drawing in the three-dimensional space is $O(n^{4/3})$. Chazelle, Edelsbrunner, Guibas, Hershberger, Seidel and Sharir [4] showed that the maximum number of edges of an n-vertex graph that has a Gabriel drawing in d-dimensional space ($d \geq 3$) is $\Omega(n^2)$.

Matula and Sokal [26] gave a partial characterization of trees that admit Gabriel drawings. In particular, they proved that every tree with vertex degree at most three admits a Gabriel drawing, while no tree with vertex degree greater than six does. Urquhart [35] gave the same two bounds on the vertex degree of relative neighborhood drawable trees. Cimikowski [5] extended further the bounds to both modified Gabriel drawable and relatively closest drawable trees. The proof given by Matula and Sokal of the Gabriel drawability of trees with maximum degree at most three gives rise to a linear time drawing algorithm.

Matula and Sokal also conjectured that Gabriel trees cannot have vertices of degree greater than four and cannot have two adjacent vertices of degree four. The truth of these conjectures was established in [3], where the proximity-drawability of trees was characterized for relative neighborhood drawings, relatively closest drawings, modified Gabriel drawings, and Gabriel drawings. In the same paper, linear time algorithms to test whether a tree admits one of the above proximity drawings were presented; furthermore, it was shown that if such a drawing exists, one can be constructed in linear time using the real-RAM model. All of the characterizations are given in terms of families of forbidden subtrees.

Open and closed strip drawings of trees were investigated in [2]. Closed strip drawable trees were completely characterized and a linear time drawing algorithm was given. In the same paper it was shown that a graph admits a closed strip drawing if and only if it is a binary forest other than one of the following: two non-adjacent vertices, a vertex and a non-adjacent edge, or two non-adjacent edges. Also in this paper, the general problem of the β-drawability of trees was studied. For several classes of trees, the values of β for which those classes admit β drawings were given, along with drawing algorithms. Kirkpatrick and Radke [23]. showed that neither non-planar graphs nor triangulated planar graphs admit open strip drawings.

Lubiw and Sleumer [25] showed that all maximal outerplanar graphs admit both a relative neighborhood drawing and a Gabriel drawing. They also proved that all biconnected outerplanar graphs can be realized as relative neighborhood drawings. Both the results lead to linear time drawing algorithms in the real-RAM model. An immediate consequence of the construction techniques adopted in their work is that all maximal outerplanar graphs are both relatively closest drawable and modified Gabriel drawable. Another consequence is that all biconnected outerplanar graphs are modified Gabriel drawable.

Negative drawability results were given in [26, 33, 5]. In [26], [33], and [24] the planarity of Gabriel drawable graphs, relative neighborhood graphs, and relatively closest drawable graphs were shown, respectively. Furthermore, in [5] it was shown that a cycle with three vertices is not relatively closest drawable.

2.4 Sphere and Rectangle of Influence Drawings

Jacobson, Lipman and McMorris [20] have characterized those trees that have open and/or closed sphere of influence drawings. They showed that the trees which admit a closed sphere of influence drawing are those trees which contain a perfect matching and those which admit an open sphere of influence drawing are those which contain what they call a (P_2, P_3)-*factor* (see [16]).

In [14], the problem of deciding whether a graph has a rectangle of influence proximity drawing was investigated for several classes of graphs. In particular, a complete characterization of open rectangle of influence drawable graphs was given for cycles, wheels, outerplanar graphs, and triangle-free graphs. As for the closed rectangle of influence drawable graphs, a complete characterization for cycles, wheels and trees is given. Also a partial characterization of maximal outerplanar graphs is presented.

References

1. P. K. Agarwal and J. Matoušek. Relative Neighbourhood Graphs in Three Dimensions. *Computational Geometry: Theory and application*, **2**, 1992, pp. 1-14.
2. P. Bose, G. Di Battista, W. Lenhart, and G. Liotta. Proximity Constraints and Representable Trees. *Proc. of GD94*.
3. P. Bose, W. Lenhart, and G. Liotta. Characterizing Proximity Trees. To appear in *Algorithmica: Special Issue on Graph Drawing*, also available as Tech. Rep. no. SOCS 93.9, School of Computer Science, McGill University.
4. B. Chazelle, H. Edelsbrunner, L. J. Guibas, J. E. Hershberger, R. Seidel, and M. Sharir. Slimming Down by Adding; Selecting Heavily Covered Points. *Proc. ACM Symp. on Comp. Geom.*, 1990, pp. 116-127.
5. R. J. Cimikowski. Properties of Some Euclidean Proximity Graphs. *Pattern Recognition Letters*, **13**, 1992, pp. 417-423.
6. M. de Berg, S. Carlsson, and M.H. Overmars. A General Approach to Dominance in the Plane. *Journal of Algorithms*, **13**, 1992, pp. 274-296.
7. B. Delaunay. Sur la Sphere vide. *Bull. Acad. Sci. USSR(VII)*, **7**, 1934, pp. 793-800.
8. G. Di Battista, P. Eades, R. Tamassia and I.G. Tollis. Algorithms for Automatic Graph Drawing: An Annotated Bibliography. To appear in *Computational Geometry: Theory and Applications*.
9. G. Di Battista and L. Vismara. Angles of Planar Triangular Graphs. *Proc. ACM Symposium on Theory of Computing*, 1993, pp. 431-437.
10. M. B. Dillencourt. Realizability of Delaunay Triangulations. *Information Proc. Letters*, **33**, 1990, pp. 283-287.
11. M. B. Dillencourt. Toughness and Delaunay Triangulations. *Discrete and Computational Geometry*, **5**, 1990, pp.575-601.
12. M. B. Dillencourt and W. D. Smith. Graph-Theoretical Conditions for Inscribability and Delaunay Realizability. *Proc. CCCG '94*, 1994, pp. 287-292.
13. P. Eades and S. Whitesides. The Realization Problem for Euclidean Minimum Spanning Trees is NP-hard. *Proc. ACM Symposium on Computational Geometry*, 1994, pp. 49-56.
14. H. ElGindy, G. Liotta, A. Lubiw, H. Meijer, and S.H. Whitesides. Recognizing Rectangle of Influence Drawable Graphs. *Proc. of GD94*.
15. K. R. Gabriel and R. R. Sokal. A New Statistical Approach to Geographical Analysis. *Systematic Zoology*, **18**, 1969, pp. 54-64.
16. F. Harary. *Graph Theory*. Addison-Wesley Pub. Comp.1969.
17. M. Garey and D. Johnson. A Guide to the Theory of NP-Completeness. W. H. Freeman and Co., New York, 1979.
18. R.H. Guting, O. Nurmi, and T. Ottmann. Fast Algorithms for Direct Enclosures and Direct Dominances. *Journal of Algorithms*, **10**, 1989, pp.170-186.
19. M. Ichino, J. Sklansky. The Relative Neighborhood Graph for Mixed Feature Variables. *Pattern Recognition*, **18**, **2**, 1985, pp. 161-167.
20. M. S. Jacobson, M. J. Lipman, and F. R. McMorris. Trees that are Sphere-of-Influence Graphs. University of Louisville, Tech. Rep. 0191/2, 1989.
21. J. W. Jaromczyk and G. T. Toussaint. Relative Neighborhood Graphs and Their Relatives. *Proceedings of the IEEE*, **80**, 1992, pp. 1502-1517.

22. J. M. Keil Computing a subgraph of the minimum weight triangulation. *Computational Geometry: Theory and Applications*, **4**, 1994, pp. 13-26.
23. D. G. Kirkpatrick and J. D. Radke. A Framework for Computational Morphology. *Computational Geometry, G. T. Toussaint*, Elsevier, Amsterdam, 1985, pp. 217-248.
24. P. M. Lankford. Regionalization: Theory and Alternative Algorithms. *Geographical Analysis*, 1, 1969, pp. 196-212.
25. A. Lubiw, and N. Sleumer, All Maximal Outerplanar Graphs are Relative Neighborhood Graphs. *Proc. CCCG '93*, 1993, pp. 198-203.
26. D. W. Matula and R. R. Sokal. Properties of Gabriel Graphs Relevant to Geographic Variation Research and the Clustering of Points in the Plane. *Geographical Analysis*, **12**, **3**, 1980, pp. 205-222.
27. C. Monma and S. Suri. Transitions in Geometric Minimum Spanning Trees. *Proc. ACM Symposium on Computational Geometry*, 1991, pp. 239-249.
28. M. S. Paterson and F.F. Yao, On Nearest-Neighbor Graphs. *Proc. ICALP '92*, 1992, pp. 416-426.
29. M. H. Overmars and D. Wood. On Rectangular Visibility. *Journal of Algorithms*, **9**, 1988, pp. 372-390.
30. J. D. Radke. On the shape of a set of points. *Computational Morphology*, ed. G. T. Toussaint, Elsevier, Amsterdam, 1988, pp. 105-136.
31. T.-H. Su and R.-Ch. Chang. The k-Gabriel Graphs and their Applications. *Proc. International Symposium, SIGAL '90*, 1990, pp. 66-75.
32. T.-H. Su and R.-Ch. Chang. Computing the k-relative neighborhood graphs in Euclidean Plane. *Pattern Recognition*, **24**, 1991, pp.231-239.
33. G. T. Toussaint. The Relative Neighborhood Graph of a Finite Planar Set. *Pattern Recognition*, **12**, 1980, pp. 261-268.
34. G. T. Toussaint. A Graph-Theoretical Primal Sketch. *Computational Morphology*, ed. G. T. Toussaint, Elsevier, Amsterdam, 1988, pp. 229-260.
35. R. B. Urquhart. Some Properties of the Planar Euclidean Relative Neighbourhood Graph. *Pattern Recognition Letters*, 1, 1983, pp. 317-322.
36. R. C. Vetkamp. The γ-Neighbourhood Graph. *Computational Geometry: Theory and Applications*, 1, 1992, pp. 227-246.